SOLUTIONS OF EXERCISES.

174

FIND the locus of the intersection of the altitudes of a triangle whose base and area are given.

SOLUTION.

Let the extremities of the base be $(\pm c, 0)$ and the vertex (x, h).

Then the declivity of the side is

$$m = \frac{h}{x - c},$$

and the declivity of the perpendicular to it

$$m' = \frac{y}{x + c}.$$

The condition of perpendicularity gives

$$hy = c^2 - x^2,$$

which represents a parabola with vertical axis passing through the extremities of the base. [*H. B. Newson; T. U. Taylor.*]

180

FIND the envelope of a system of circles each of which is seen from two fixed points under a constant angle.

SOLUTION.

Let the line joining the fixed points be the x-axis and its middle point be the origin. Let $2c$ be the distance between the points; let $\sin^{-1}(m)$ be half the constant angle.

It is evident that the locus of the centres of the system is the y-axis. Let (o, a) be the co-ordinates of the centre of any circle of the system. Then

$$r^2 = (a^2 + c^2) m^2,$$

and the equation to the circle is

$$x^2 + (y - a)^2 = (a^2 + c^2) m^2.$$

Differentiating with respect to the parameter a, we obtain

$$-2(y - a) = 2am^2;$$

whence $$a = \frac{y}{1 - m^2}.$$

This gives for the equation to the envelope

$$\frac{x^2}{m^2} - \frac{y^2}{1 - m^2} = c^2,$$

which represents an hyperbola with the fixed points as foci and $e = m^{-1}$.

[*H. B. Newson.*]

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SUPPOSE a vessel in the form of an inverted frustum of a cone to be filled with water, and then tipped until as much water runs out as is possible without uncovering any part of the bottom; required, the volume of the water which remains.

SOLUTION.

Let R, r be the radii of the bases of the frustum, H altitude, a, b semi axis of the elliptic water surface. The supplemental cone has a volume

$$v = \tfrac{1}{3}\pi H \frac{r^3}{R - r}.$$

This added to the required volume gives an oblique cone whose altitude is

$$k = \frac{2hRr}{a(R - r)},$$

and whose volume, therefore, is

$$\tfrac{1}{3}\pi abk = \tfrac{1}{3}\pi H . \frac{b^3}{R - r}.$$

Hence the required volume is

$$V = \tfrac{1}{3}\pi H . \frac{b^3 - r^3}{R - r},$$

in which $$b = \sqrt{Rr},$$

as may be easily proved. [*B. E. Sheppard.*]

Also solved by *Mr. O. L. Mathiot.*

230

SOLVE the equations

$$x^2(y - z) = a,$$
$$y^2(z - x) = b,$$
$$z^2(x - y) = c.$$

[*Frank Morley.*]

SOLUTION.

$$a + b + c = -(y - z)(z - x)(x - y),$$
$$abc = x^2 y^2 z^2 (y - z)(z - x)(x - y),$$

whence
$$xyz = k$$
is known.

Accordingly
$$ayz = k(xy - zx),$$
$$bzx = k(yz - xy);$$
whence
$$ayz + bzx = k(yz - zx),$$
or
$$\frac{x}{k-a} = \frac{y}{k+b},$$
$$\frac{y}{k-b} = \frac{z}{k+c},$$
$$\frac{z}{k-c} = \frac{x}{k+a},$$

are obtained in the same way, By means of these relations we eliminate y and z (say) from the first of the given equations and find x; thus

$$x^3\left[\frac{k+b}{k-a} - \frac{k-c}{k+a}\right] = a.$$

The value of k is
$$k = \sqrt{\frac{-abc}{a+b+c}}$$

[*T. U. Taylor.*]

231

If we take products of n consecutive terms of the arithmetical series a, $a - d$, etc., commencing for the first product with the first term, for the second product with the second term, and so on; and then multiply these products by the coefficients in the expansion of $(1 - x)^n$, the aggregate will be $n!d^n$, which is independent of the first term. [*W. W. Johnson.*]

SOLUTION.

If the n differences of a series of $n + 1$ quantities be multiplied by the coefficients in the expansion of $(1 - x)^{n-1}$, the aggregate is readily seen to be the same as that of the $n + 1$ quantities multiplied by the coefficients of $(1 - x)^n$.

Now the first product named in the question is

$$a(a - d)\ldots[a - (n - 1)d],$$

and the second is

$$(a - d)\ldots[a - (n - 1)d](a - nd).$$

Thus their difference is

$$nd \times (a - d)\ldots[a - (n - 1)d].$$

Considering in like manner the differences between each pair of consecutive products, the theorem above shows that the aggregate in question is the product of nd into a similar aggregate in which n is changed to $n-1$, and d is unchanged. Thus denoting the aggregate by S_n, we have

$$S_n = nd\,.\,S_{n-1} = nd\,(n-1)\,d\,.\,S_{n-2} = \ldots = n!\,d^n,$$

since $S_1 = d$. [*W. W. Johnson.*]

Also solved by *Dr. R. A. Harris.*

234

In a regular heptagon $ABCDEFG$, show that

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

[*Frank Morley.*]

SOLUTION.

Let $x = \frac{1}{7}\pi$; then

$$\frac{1}{\sin x} - \frac{1}{\sin 3x} = \frac{2\cos 2x \sin x}{\sin x \sin 3x}$$

$$= \frac{2\cos 2x}{\sin 4x}$$

$$= \frac{1}{\sin 2x},$$

which was to be proved. [*T. U. Taylor.*]

Also solved by *Dr. R. A. Harris.*

235

Find the areas of the greatest and of the least rhombus inscribed in an ellipse. [*R. H. Graves.*]

SOLUTION I.

The central radius vector of an ellipse is given by the formula

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

$$= \frac{1}{b^2}(1 - e^2\cos^2\theta).$$

If r_1, r_2 be orthogonal radii, the area of the rhombus of which they are semi-diagonals is

$$2r_1r_2 = \frac{2b^2}{\sqrt{[(1-e^2\cos^2\theta)(1-e^2\sin^2\theta)]}}.$$

This has its least value when

$$(1-e^2\cos^2\theta)(1-e^2\sin^2\theta) = z$$

is greatest, and its greatest value when this is least. But

$$z = 1 - e^2 + \tfrac{1}{4}e^4 \sin^2 2\theta$$

is greatest when $\sin 2\theta = 1$, and least when $\sin 2\theta = 0$. The corresponding values of the area are

$$\Omega' = \frac{2b^2}{1 - \frac{1}{2}e^2}, \qquad \Omega'' = \frac{2b^2}{\sqrt{1 - e^2}};$$

or

$$\Omega' = \frac{4a^2b^2}{a^2 + b^2}, \qquad \Omega'' = 2ab.$$

[*W. M. Thornton.*]

SOLUTION II.

Let the ellipse be projected into a circle whose diameter is equal to the minor axis. Every parallelogram inscribed in the ellipse becomes a rectangle inscribed in the circle. In case of the rhombus, the sides being as $1 : 1$, after projection the ratio of two adjacent sides of the rectangle may vary from 1 to $\frac{b}{a}$. Find now the greatest and least rectangles the ratios of whose adjacent sides lie within these limits.

The first is a square. For, draw axes through the centre of the rectangle parallel to its sides (the x-axis being parallel to the longer side), and let θ denote the angle which the diagonal makes with the x-axis. Then the area is

$$u = b^2 \sin\theta \cos\theta.$$

$$\therefore \frac{du}{d\theta} = b^2 \cos 2\theta;$$

and $\theta = 45^\circ$ causes $\frac{du}{d\theta}$ to vanish.

We can project this square into a rhombus only by projecting the diagonals into the major and minor axes. Hence, the area of the greatest rhombus is $2ab$.

u being an increasing function up to $\theta = 45^\circ$, the smaller we can take θ, the smaller will be the area. This value is $\theta = \tan^{-1}\left(\frac{b}{a}\right)$; i. e. the ratio of the two adjacent sides of our rectangle is $\frac{b}{a}$. Hence, the length of its longer side is $\frac{2ab}{\sqrt{a^2 + b^2}}$.

We can project this rectangle into a rhombus only by projecting the shorter sides into lines parallel to the major axis; the longer sides into lines parallel to the minor. This rhombus, then, is a square whose area is $\frac{4a^2b^2}{a^2 + b^2}$.

[*R. A. Harris.*]